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The Gurevich–Zybin system

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Abstract
We present three different linearizable extensions of the Gurevich–Zybin system. Their general solutions are found by reciprocal transformations. In this paper we rewrite the Gurevich–Zybin system as a Monge–Ampere equation. By application of reciprocal transformation this equation is linearized. Infinitely many local Hamiltonian structures, local Lagrangian representations, local conservation laws and local commuting flows are found. Moreover, all commuting flows can be written as Monge–Ampere equations similar to the Gurevich–Zybin system. The Gurevich–Zybin system describes the formation of large scale structure in the Universe. Second harmonic wave generation is known in nonlinear optics. In this paper we prove that the Gurevich–Zybin system is equivalent to a degenerate case of second harmonic generation. Thus, the Gurevich–Zybin system is recognized as a degenerate first negative flow of two-component Harry Dym hierarchy up to two Miura-type transformations. A reciprocal transformation between the Gurevich–Zybin system and degenerate case of the second harmonic generation system is found. A new solution for second harmonic generation is presented in implicit form.

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To the memory of Professor Andrea Donato (Messina University)

1. Introduction

Invisible nondissipative dark matter plays a decisive role in the formation of large scale structure in the Universe: galaxies, clusters of galaxies, superclusters. Corresponding nonlinear dynamics can be described (see [1]) by the following hydrodynamic-like system,

$$\rho_t + \nabla(\rho \mathbf{u}) = 0, \qquad u_t + (\mathbf{u}\nabla)u + \nabla\Phi = 0, \qquad \Delta\Phi = \rho, \tag{1}$$

where the first two equations are the usual hydrodynamic equations (the continuity equation and the Euler equation, respectively), but the third is the famous Poisson equation. This system was first derived by Jeans (see [2] and also [3]) for a description of instabilities of a homogeneous distribution of a matter.

Such dynamics of a dissipationless gravitating gas is a special limit ($\varepsilon \to 0$) of another system (another sign of ρ is inessential)

$$\rho_t + \nabla(\rho\mathbf{u}) = 0, \qquad u_t + (\mathbf{u}\nabla)u + \nabla\Phi = 0, \qquad \Delta\Phi = \varepsilon\,\mathrm{e}^{\Phi} - \rho$$

describing fully nonlinear flows in a two-temperature unmagnetized collisionless plasma in dimensionless variables (nonlinear ion-acoustic waves, see for instance [4]).

The main advantage of the Jeans theory is a reckoning of two factors: gravity attracting matter in separate lumps and clots, and pressure decreasing an inhomogeneity of matter in the Universe.

Recently a new achievement in the investigation of system (1) was made (see [1]) in cosmology. The nonlinear one-dimensional dynamics of dark matter is described by the equations [1]

$$u_t + uu_x + \upsilon = 0, \qquad \upsilon_t + u\upsilon_x = 0, \tag{2}$$

where $\rho = \upsilon_x$, $\upsilon = \Phi_x$. The analysis of equations (2) in multimode form demonstrates the transition from the hydrodynamic to the equilibrium kinetic state [1]. It means that the exact solution of equations (2) describes a fundamental physical process (see [1]).

It is amazing that the *inhomogeneous* hydrodynamic-type system (2) can be integrated, up to the first singularity, by the hodograph method (see [1]). For this reason we will henceforth call system (2) the Gurevich–Zybin system, emphasizing that the one-dimensional reduction (2) of the system (1) is integrable.

Here we give the general solution by the method of *reciprocal transformations*. Moreover, we present three different linearizable extensions of this system (2) with their general solutions given by corresponding reciprocal transformations. Actually these reciprocal transformations have clear pure mathematical (hodograph method) and physical (transition from Euler to Lagrange variables) interpretations. In the next section we present three linearizable extensions of the Gurevich–Zybin systems with their general solutions. In the third section the relationship between two-component generalization of the Hunter–Saxton equation and the Gurevich–Zybin system is established. In the fourth section the Gurevich–Zybin system is rewritten as a Monge–Ampere equation (following the approach developed by Andrea Donato). In the fifth section a bi-Hamiltonian structure of the Gurevich–Zybin system is found (following the approach developed by Yavuz Nutku). In the sixth section, by the application of a reciprocal transformation, the simplest recursion operator is constructed. Infinitely many local conservation laws, local commuting flows, local Lagrangians and local Hamiltonians are found. Moreover, all commuting flows are Monge–Ampere equations. Thus, the Gurevich–Zybin system is a member of an integrable hierarchy of Monge–Ampere equations. In the seventh section a bi-Hamiltonian formulation for the Gurevich–Zybin system is given in a canonical form. The Gurevich–Zybin system is recognized as a first negative flow of two-component Harry Dym hierarchy. In the eighth section Miura type and reciprocal transformations between the Gurevich–Zybin system and Kaup–Boussinesq hierarchy are given. In the ninth section we finally prove that the Gurevich–Zybin system is equivalent to a degenerate case of the second harmonic generation system up to the above-mentioned transformations. A new solution of the second harmonic generation system is found. In the tenth section we discuss integrability of the Gurevich–Zybin system in the *N*-component case. In conclusion we discuss the sort of integrable problems belonging to some different hierarchies of integrable equations.

2. General solution

The Gurevich–Zybin system (1) in the one-dimensional case precisely has the form

$$\rho_t + \partial_x(\rho u) = 0, \qquad u_t + uu_x + \Phi_x = 0, \qquad \Phi_{xx} = \rho. \tag{3}$$

This system can be generalized in *at least* three different forms:

$$\rho_t + \partial_x(\rho u) = 0, \qquad u_t + uu_x + \mu'''(\Phi_x) = 0, \qquad \Phi_{xx} = \rho, \tag{4}$$

$$\rho_t + \partial_x(\rho u) = 0, \qquad u_t + uu_x + \Phi_x = 0, \qquad \Phi_{xx} = B(\rho), \tag{5}$$

$$\rho_t + \partial_x(\rho u) = 0, \qquad u_t + uu_x + \mu'''(\Phi_x) = 0, \qquad \partial_x C(\Phi_x) = \rho, \tag{6}$$

where $\mu(z)$, $B(\rho)$ and $C(z)$ are arbitrary functions. It is interesting that it is not obvious that system (4) is equivalent to the Gurevich–Zybin system (3). Indeed, the system (4) written like (2)

$$z_t + uz_x = 0, \qquad u_t + uu_x + \mu'''(z) = 0 \tag{7}$$

is exactly (2) up to the point transformation $\upsilon = \mu'''(z)$. One can introduce the reciprocal transformation

$$\mathrm{d}z = \rho\,\mathrm{d}x - \rho u\,\mathrm{d}t, \qquad \mathrm{d}\tau = \mathrm{d}t. \tag{8}$$

Then $\partial_x = \rho\partial_z$ and $\partial_t = \partial_\tau - \rho u\partial_z$. Thus, the system (4) has the form

$$\left(\frac{1}{\rho}\right)_\tau = u_z, \qquad u_\tau = -\mu'''(z), \tag{9}$$

in new variables, where $z = \Phi_x$. Thus, the general solution of system (9) is

$$u = -\mu'''(z)\tau + D'(z), \qquad \frac{1}{\rho} = -\mu''''(z)\tau^2/2 + D''(z)\tau + E''(z),$$

where $D(z)$ and $E(z)$ are arbitrary functions. Finally, the general solution of system (4) can be given in the implicit form

$$\begin{aligned} &u = -\mu'''(z)t + D'(z), \qquad \rho = [-\mu''''(z)t^2/2 + D''(z)t + E''(z)]^{-1}, \\ &x = -\mu'''(z)t^2/2 + D'(z)t + E'(z), \\ &\Phi = (\mu''(z) - z\mu'''(z))t^2/2 + (zD'(z) - D(z))t + zE'(z) - E(z), \end{aligned} \tag{10}$$

where z is a parameter here.

The above reciprocal transformation applied to system (5) yields

$$\left(\frac{1}{\rho}\right)_\tau = u_z, \qquad u_\tau = -\upsilon, \qquad \upsilon_z = B(\rho)/\rho,$$

where $\upsilon = \Phi_x$. If the function ρ can be explicitly expressed from the algebraic equation

$$\tau = G(z) - \int^\rho \frac{\mathrm{d}\theta}{\theta^2\sqrt{2\int B(\theta)\theta^{-3}\,\mathrm{d}\theta - F(z)}},$$

where $F(z)$ and $G(z)$ are arbitrary functions, then the general solution of system (5) can be obtained. For instance, if $B(\rho) = \rho$, then a general solution is already given by (10) (remember that in such sub-case systems (4) and (5) coincide if $\mu(z) = z^4/24$); in the simplest perturbed case $B(\rho) = \rho + \delta/\rho$ ($\delta = \text{const}$) the general solution is expressed via Weierstrass elliptic functions

$$\rho = \frac{\delta}{6}\left[\wp\left(\tau - G(z), \frac{\delta}{3}, \frac{\delta^2}{36}F(z)\right)\right]^{-1}, \qquad u_z = \frac{6}{\delta}\wp', \qquad \upsilon_z = 1 - \frac{36}{\delta}\wp^2.$$

The reciprocal transformation (8) applied to the system (6) yields

$$\left(\frac{1}{\rho}\right)_\tau = u_z, \qquad u_\tau = -\mu'''(\upsilon), \qquad C(\upsilon) = z,$$

where $\upsilon = \Phi_x$. Thus, the general solution of this system is

$$\upsilon = V(z), \qquad u = D'(z) - \mu'''(\upsilon)\tau,$$
$$\rho = \left[E''(z) + D''(z)\tau - \frac{\mu''''(\upsilon)}{2C'(\upsilon)}\tau^2\right]^{-1},$$
$$x = E'(z) + D'(z)\tau - \mu'''(\upsilon)\tau^2/2,$$

where $D(z)$, $E(z)$ are arbitrary functions and $V(z)$ is the inverse function to $C(\upsilon)$. Finally, the general solution of system (6) can be given in the implicit form

$$\upsilon = V(z), \qquad u = D'(z) - \mu'''(\upsilon)t,$$
$$\rho = \left[E''(z) + D''(z)t - \frac{\mu''''(\upsilon)}{2C'(\upsilon)}t^2\right]^{-1},$$
$$x = E'(z) + D'(z)t - \mu'''(\upsilon)t^2/2,$$
$$\Phi = zE'(z) - E(z) + (zD'(z) - D(z))t + [G(\upsilon) - zG'(\upsilon)]t^2/2,$$

where $G' = \mu'''C'$ and z is a parameter here. Also, the system (6) can be written in the hydrodynamic-like form

$$\rho_t + \partial_x(\rho u) = 0, \qquad u_t + uu_x + \frac{1}{\rho}\partial_x P = 0, \tag{11}$$

where the pressure P is a nonlocal function of the density ρ

$$P = P\big(V\big(\partial_x^{-1}\rho\big)\big).$$

In the particular case $C(\upsilon) = \upsilon$ the system (6) coincides with the system (4); the system (6) was written in form (11) in [5] for the particular case $\mu(\upsilon) = \upsilon^4/24$ and $C(\upsilon) = \upsilon$.

3. Two-component generalization of the Calogero equation

The simplest two-component linearizable generalization

$$\eta_t + \partial_x(\eta u) = 0, \qquad u_{xt} + uu_{xx} + \Psi(\eta, u_x) = 0 \tag{12}$$

of the Calogero equation (see [6])

$$u_{xt} = uu_{xx} + R(u_x)$$

was presented (functions $\Psi(a, b)$ and $R(c)$ are arbitrary here) in [7]. Some particular cases of the Calogero equation such as the Hunter–Saxton equation (i.e. $R(c) = \nu c^2$, where $\nu = \text{const}$; see [8]) are interesting from a physical point of view. The Calogero equation is linearizable by a reciprocal transformation (see [7]). For instance, the Hunter–Saxton equation is related to the famous Liouville equation by a reciprocal transformation (see [9]). Thus, the system (12) is a natural generalization of the Liouville equation on the two-component case up to a module of a (invertible) reciprocal transformation.

The reciprocal transformation

$$\mathrm{d}\zeta = \eta\,\mathrm{d}x - \eta u\,\mathrm{d}t, \qquad \mathrm{d}y = \mathrm{d}t \tag{13}$$

applied to system (12) yields the *ordinary* differential equation

$$s_{yy} + \Psi(\mathrm{e}^{-s}, s_y) = 0,$$

where $s = -\ln\eta$, which can be reduced to the first-order equation

$$a\,\mathrm{d}a + \Psi(\mathrm{e}^{-s}, a)\,\mathrm{d}s = 0, \tag{14}$$

where $a = s_y = u_x$. Then a general solution can be constructed in two steps from

$$u_\zeta = \left(\frac{1}{\eta}\right)_y, \qquad \mathrm{d}x = \frac{1}{\eta}\,\mathrm{d}\zeta + u\,\mathrm{d}y.$$

A solution $q(s, a)$ of the linear equation

$$\frac{\partial q}{\partial s} = \frac{\Psi(\mathrm{e}^{-s}, a)}{a}\frac{\partial q}{\partial a} \tag{15}$$

determined by the characteristic equation (14) yields the extra conservation law

$$\rho_t + \partial_x(\rho u) = 0,$$

where $\rho(\eta, u_x) = \eta \exp q$. The comparison of the second equation in (12) and the second equation in (3) yields another relationship

$$\Psi = \rho + u_x^2.$$

Thus, a solution of the *nonlinear* equation (substitute Ψ from the above equation into (15))

$$q_s = \left(a + \frac{\mathrm{e}^{q-s}}{a}\right) q_a$$

describes a transformation between the Gurevich–Zybin system (3) and the two-component generalization of the Hunter–Saxton equation (12).

Remark 1. The above equation under the substitution

$$q = s + \ln\left(\frac{n}{2}\,\mathrm{e}^{-3s} - a^2\right)$$

transforms into the well-known inhomogeneous Riemann–Monge–Hopf equation

$$n_y + n n_c = -\frac{2c}{9y^2},$$

where $y = \mathrm{e}^{-3s}/3$ and $c = a^2$. Its general solution can be given just in the parametric form

$$n = \tfrac{1}{3}A_1(\xi)y^{-2/3} + \tfrac{2}{3}A_2(\xi)y^{-1/3}, \qquad c = A_1(\xi)y^{1/3} + A_2(\xi)y^{2/3},$$

where ξ is a parameter and $A_1(\xi)$, $A_2(\xi)$ are arbitrary functions. However, the general solution of the equation of the first order depends on one function of a single variable only. It means that if for instance $A_1(\xi) \neq \text{const}$, then by re-scaling $A_1(\xi) \to \xi$ the general solution takes the form

$$n = \tfrac{1}{3}\xi y^{-2/3} + \tfrac{2}{3}A(\xi)y^{-1/3}, \qquad c = \xi y^{1/3} + A(\xi)y^{2/3},$$

where $A(\xi)$ is an arbitrary function.

In a particular case the substitutions (see [7])

$$\rho = -\tfrac{1}{2}\left(u_x^2 + \eta^2\right), \qquad \Psi(\eta, u_x) = \tfrac{1}{2}\left(u_x^2 - \eta^2\right) \tag{16}$$

connect the Gurevich–Zybin system (3) with two-component Hunter–Saxton system (see (12) and [10]), which is a *bi-directional* version of the Hunter–Saxton equation. A general solution of the Gurevich–Zybin system in field variables η and u has the implicit form with respect to the parameter ζ

$$\eta = \left[\frac{1}{k'(\zeta)} + \frac{1}{4}k'(\zeta)(t - m(\zeta))^2\right]^{-1},$$
$$u = \frac{1}{2}tk(\zeta) - \frac{1}{2}\int m(\zeta)\,\mathrm{d}k(\zeta), \tag{17}$$
$$x = \int\left[\frac{1}{k'(\zeta)} + \frac{1}{4}k'(\zeta)m^2(\zeta)\right]\mathrm{d}\zeta + \frac{1}{4}t^2k(\zeta) - \frac{1}{2}t\int m(\zeta)\,\mathrm{d}k(\zeta),$$

where $m(\zeta), k(\zeta)$ are arbitrary functions. One can substitute the above expression for η into the first equation in (16) and compare expressions for (ρ, x, u) from (10) (in this case $\mu'''(z) = z$) and above; then a relationship between the arbitrary functions k, m and D, E will be reconstructed.

4. Reformulation of the Gurevich–Zybin system as a Monge–Ampere equation

A lot of physically motivated nonlinear systems can be written as Monge–Ampere equations (see [11]). At this moment we have no unique method for constructing such relationships. One simple approach was suggested to the author by Andrea Donato (see [12]) at the 'Lie Group Analysis' conference in Johannesburg at 1996.

The Gurevich–Zybin system in physical field variables (3) has four local conservation laws

$$u_t + \partial_x(u^2/2 + \Phi) = 0, \qquad \rho_t + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x\left(\rho u^2 + \Phi_x^2/2\right) = 0, \qquad \partial_t\left(\rho u^2 - \Phi_x^2\right) + \partial_x(\rho u^3) = 0$$

as a consequence of the obvious local Hamiltonian structure

$$\upsilon_t = \frac{\delta H_2}{\delta u}, \qquad u_t = -\frac{\delta H_2}{\delta \upsilon}, \tag{18}$$

where the Hamiltonian is $H_2 = \frac{1}{2}\int[-u^2\upsilon_x + \upsilon^2]\,\mathrm{d}x$, the momentum is $H_1 = \int u\upsilon_x\,\mathrm{d}x$, and two Casimirs are functionals $Q_1 = \int u\,\mathrm{d}x$ and $Q_2 = \int \rho\,\mathrm{d}x$ of the corresponding Poisson bracket

$$\{\rho(x), u(x')\} = \{u(x), \rho(x')\} = \delta'(x - x'). \tag{19}$$

The existence of the above first three local conservation laws is obvious. However, the fourth conservation law is not easy to check. Since $\rho = \Phi_{xx}$, then $\rho u = -\Phi_{xt}$, then $\rho u^2 + \Phi_x^2/2 = \Phi_{tt}$, then the above fourth conservation law is valid. Eliminating physical field variables ρ and u from these three equations, the Monge–Ampere equation is given by

$$\Phi_{xx}\Phi_{tt} - \Phi_{xt}^2 = \tfrac{1}{2}\Phi_x^2\Phi_{xx}. \tag{20}$$

In paper [1] the Gurevich–Zybin system was linearized by a hodograph method. A general solution has also been presented. Thus, this Monge–Ampere equation is linearizable and has the general solution in implicit form (see the end of section 2)

$$\Phi = -\tfrac{1}{4}\upsilon^2t^2 + (\upsilon D'(\upsilon) - D(\upsilon))t + \upsilon E'(\upsilon) - E(\upsilon), \qquad x = -\upsilon t^2/2 + D'(\upsilon)t + E'(\upsilon),$$

where $D(\upsilon)$ and $E(\upsilon)$ are arbitrary functions, υ is a parameter here.

Since the Gurevich–Zybin system can be written in the form (4), (7), we shall use an arbitrary value of function $\mu(z)$ in the next two sections.

5. Bi-Hamiltonian structure

The Gurevich–Zybin system (7) has local bi-Hamiltonian structure, where the first local Hamiltonian structure is (18)

$$z_t = \frac{\delta H_2}{\delta u}, \qquad u_t = -\frac{\delta H_2}{\delta z}. \tag{21}$$

There are just three conservation laws

$$\partial_t \rho + \partial_x(\rho u) = 0, \qquad \partial_t(\rho u) + \partial_x[\rho u^2 + \mu''(\Phi_x)] = 0,$$
$$\partial_t[\rho u^2 - 2\mu''(\Phi_x)] + \partial_x(\rho u^3) = 0,$$

associated with the first Poisson bracket (19). The Hamiltonian is $H_2 = \int[-\rho u^2 + 2\mu''(\Phi_x)]\,\mathrm{d}x$, the momentum is $H_1 = \int \rho u\,\mathrm{d}x$ and the Casimir is $H_0 = \int \rho\,\mathrm{d}x$. The corresponding Lagrangian representation is

$$S_1 = \int \left[\frac{z_t^2}{2z_x} + \mu''(z)\right] \mathrm{d}x\,\mathrm{d}t, \tag{22}$$

where $u = -z_t/z_x$. Thus, the Lagrangian

$$S_1 = \int \left[\frac{\Phi_{xt}^2}{2\Phi_{xx}} + \mu''(\Phi_x)\right] \mathrm{d}x\,\mathrm{d}t$$

creates the Euler–Lagrange equation

$$\Phi_{xx}\Phi_{tt} - \Phi_{xt}^2 = \mu''(\Phi_x)\Phi_{xx}, \tag{23}$$

which is a Monge–Ampere equation (cf (20)).

At the same time (23) allows another Lagrangian representation (see [13])

$$S_2 = \int \left[\frac{1}{2}\Phi_{xx}\Phi_t^2 - \mu(\Phi_x)\right] \mathrm{d}x\,\mathrm{d}t. \tag{24}$$

Thus, the Monge–Ampere equation (23) has the second Hamiltonian structure

$$r_t = \partial_x \frac{\delta \bar{H}_2}{\delta z}, \qquad z_t = \partial_x \frac{\delta \bar{H}_2}{\delta r}$$

determined by the local Poisson bracket

$$\{z(x), r(x')\}_2 = \{r(x), z(x')\}_2 = \delta'(x - x'), \tag{25}$$

where $r = \Phi_{xx}\Phi_t$, the Hamiltonian is $\bar{H}_2 = \int\left[\frac{r^2}{2z_x} + \mu(z)\right]\mathrm{d}x$, the momentum is $\bar{H}_1 = \int rz\,\mathrm{d}x$, two Casimirs are $\bar{Q}_1 = \int r\,\mathrm{d}x$ and $\bar{Q}_2 = \int z\,\mathrm{d}x$. Four local conservation laws associated with the above second Hamiltonian structure are

$$z_t = \partial_x\left(\frac{r}{z_x}\right), \qquad \partial_t\left[\frac{r^2}{2z_x} + \mu(z)\right] = \partial_x\left[\mu'(z)\frac{r}{z_x} + \frac{1}{6}\partial_x\left(\frac{r^3}{z_x^3}\right)\right],$$
$$r_t = \partial_x\left[\mu'(z) + \partial_x\left(\frac{r^2}{2z_x^2}\right)\right], \qquad \partial_t(rz) = \partial_x\left[z\mu'(z) - \mu(z) + \frac{z}{2}\partial_x\left(\frac{r^2}{z_x^2}\right)\right].$$

6. Recursion operator: integrability of the GZ hierarchy

Applying the reciprocal transformation (8) simultaneously to both Lagrangian representations (22) and (24), one obtains the variation principles in *other* independent variables (a recalculation of Lagrangians under reciprocal transformations is given in detail in [14])

$$S_1 = \int \left[\frac{1}{2}x_\tau^2 + \mu''(z)x_z\right] \mathrm{d}z\,\mathrm{d}\tau \qquad \text{and} \qquad S_2 = \int \left[\frac{1}{2}\tilde{\Phi}_\tau^2 - \mu(z)\tilde{\Phi}_{zz}\right] \mathrm{d}z\,\mathrm{d}\tau,$$

where $u = x_\tau$, $\rho^{-1} = x_z$, $x = \tilde{\Phi}_z$ (see the first conservation law associated with the second Hamiltonian structure (25): $\mathrm{d}\Phi = z\,\mathrm{d}x + \frac{r}{\rho}\,\mathrm{d}t$ or $\mathrm{d}\tilde{\Phi} = x\,\mathrm{d}z - \frac{r}{\rho}\,\mathrm{d}\tau$, where $xz = \Phi + \tilde{\Phi}$). Then the Euler–Lagrange equation is $x_{\tau\tau} = -\mu'''(z)$. This equation can easily be integrated (see (10)). The corresponding Poisson brackets (see (19) and (25))

$$\{x(z), u(z')\}_1 = -\{u(z), x(z')\}_1 = \delta(z - z'),$$
$$\{p(z), \tilde{\Phi}(z')\}_2 = -\{\tilde{\Phi}(z), p(z')\}_2 = \delta(z - z'),$$

where $u = \tilde{\Phi}_{z\tau}$ and $\mathrm{d}\tilde{\Phi} = x\,\mathrm{d}z - p\,\mathrm{d}\tau$ (i.e. $p = r/\rho \equiv \Phi_t$), create the recursion operator

$$\hat{R} = -\begin{pmatrix} \partial_z^2 & \\ & \partial_z^2 \end{pmatrix},$$

where

$$\{x(z), u(z')\}_2 = -\{u(z), x(z')\}_2 = -\delta''(z - z').$$

Thus, the Gurevich–Zybin system in these independent variables has an *infinite set of local Hamiltonian structures, conservation laws and commuting flows*. For instance, all such Hamiltonians are

$$\tilde{H}_k = (-1)^k \int \left[\frac{1}{2} p^{(k)^2} + \mu^{(k+2)}(z)\tilde{\Phi}^{(k)}\right] \mathrm{d}z, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

The corresponding commuting flows

$$\tilde{\Phi}_{\tau^k \tau^k} = -\mu^{(k+2)}(z)$$

can easily be integrated (see (10)). However, in the independent variables (x, t^k) they can be written in the form (cf (7))

$$\partial_{t^k} u^k + u^k \partial_x u^k + \mu^{(2k+3)}(z) = 0, \qquad \partial_{t^k} z + u^k \partial_x z = 0, \tag{26}$$

where

$$u^0 \equiv u, \qquad u^{k+1} = \frac{1}{\rho}\partial_x u^k, \qquad u^{-k-1} = \partial_x^{-1}(\rho u^{-k}), \qquad k = 0, 1, 2, \ldots. \tag{27}$$

Thus, *all commuting flows* to the Gurevich–Zybin system created by the above bi-Hamiltonian structure are *Monge–Ampere equations* (cf (23))

$$\Phi_{xx}\Phi_{t^k t^k} - \Phi_{xt^k}^2 = \mu^{(2k+2)}(\Phi_x)\Phi_{xx},$$

where

$$u^k = -\Phi_{t^{k+1}}, \qquad \Phi_{xt^k} = \Phi_{xx}\Phi_{t^{k+1}}, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

All local Lagrangians are

$$S_{2,k} = \int \left[\frac{1}{2}\Phi_{xx}\Phi_{t^k}^2 - \mu^{(2k)}(\Phi_x)\right] \mathrm{d}x\,\mathrm{d}t^k,$$

$$S_{1,k} = \int \left[\frac{\Phi_{xt^k}^2}{2\Phi_{xx}} + \mu^{(2k+2)}(\Phi_x)\right] \mathrm{d}x\,\mathrm{d}t^k,$$

$$S_{0,k} = \int \left[\frac{1}{2\Phi_{xx}}\left[\left(\frac{\Phi_{xt^k}}{\Phi_{xx}}\right)_x\right]^2 - \mu^{(2k+4)}(\Phi_x)\right] \mathrm{d}x\,\mathrm{d}t^k,$$

$$S_{-1,k} = \int \left[\frac{1}{2\Phi_{xx}}\left[\left(\frac{1}{\Phi_{xx}}\left[\left(\frac{\Phi_{xt^k}}{\Phi_{xx}}\right)_x\right]\right)_x\right]^2 + \mu^{(2k+6)}(\Phi_x)\right] \mathrm{d}x\,\mathrm{d}t^k, \ldots$$

Remark 2. All commuting flows have infinitely many different *local* representations via different pairs of field variables (z, u^k), see (27). For instance (cf (26))

$$z_{t^k} + \partial_x u^{k-1} = 0, \qquad \partial_{t^k} u^{k-1} + \frac{\left(u_x^{k-1}\right)^2}{z_x} + \mu^{(2k+2)}(z) = 0,$$

$$z_{t^k} + \partial_x \left(\frac{u_x^{k-2}}{z_x}\right) = 0, \qquad \partial_{t^k} u^{k-2} + \partial_x \left[\frac{\left(u_x^{k-2}\right)^2}{2z_x^2}\right] + \mu^{(2k+1)}(z) = 0.$$

The theory of integrable systems with a multi-Lagrangian structure is presented in [15] (see also [16]). Usually, every local Lagrangian creates a nonlocal Hamiltonian structure. Such explicit formulae of nonlocal Hamiltonian structures, nonlocal commuting flows, nonlocal conservation laws as well as nonlocal Lagrangians can be found iteratively from the formulae already given above.

7. Another bi-Hamiltonian structure

Now in this and the next two sections we identify $v \equiv z$, i.e. we concentrate on the case $\mu'''(z) = z$ (see (4)). In the two previous sections we discussed bi-Hamiltonian structure of the Gurevich–Zybin system. Here we preserve the first Hamiltonian structure (see (18), (21)), but change the second one! However, this new second Hamiltonian structure (see below) is not from this set!

The Gurevich–Zybin system (2) is an Euler–Lagrange equation of corresponding variational principle (see (22), when $\mu''(z) = z^2/2$)

$$S = \frac{1}{2} \int \left[\frac{z_t^2}{z_x} + z^2\right] \mathrm{d}x\, \mathrm{d}t.$$

However, the *astonishing* fact is that the *Gurevich–Zybin system* (2) *has* another *Hamiltonian structure connected with the* ***same*** *Lagrangian density*. Namely (see for details [16], especially formulae (43), (52)–(54) therein), the Lagrangian (cf S)

$$\tilde{S} = \frac{1}{2} \int \left[\frac{p_x}{z_x}(2z_t - p_x) + z^2\right] \mathrm{d}x\, \mathrm{d}t$$

determines the same Euler–Lagrange equations (2) but with another Hamiltonian structure

$$u_t = -\partial_x^{-1} \frac{\delta H_1}{\delta u} + u_x \partial_x^{-1} \frac{\delta H_1}{\delta z}, \qquad z_t = \partial_x^{-1} \left(u_x \frac{\delta H_1}{\delta u} + z_x \frac{\delta H_1}{\delta z}\right) + z_x \partial_x^{-1} \frac{\delta H_1}{\delta z},$$

where $u = -p_x/z_x$ (i.e. $p = \Phi_t$).

Remark 3. This bi-Hamiltonian structure first was discovered by Yavuz Nutku [17] and later it was independently found in [5] (see formula (9) therein) exactly as was done in [16]. However, here we repeat and emphasize the main observation of this section is that *both Hamiltonian structures have the* ***same*** *Lagrangian density!* This is the first such example in the theory of integrable systems.

Canonical representation for both Hamiltonian structures and recursion operator. The Poisson bracket

$$\{u(x), u(x')\}_1 = 0, \qquad \{\rho(x), u(x')\}_1 = \delta'(x - x'),$$

$$\{u(x), \rho(x')\}_1 = \delta'(x - x'), \qquad \{\rho(x), \rho(x')\}_1 = 0$$

of the first Hamiltonian structure is given in its canonical form (for more details see the review [18]). However, the Poisson bracket

$$\{u(x), u(x')\}_2 = -\partial^{-1}\delta(x-x'), \qquad \{u(x), \rho(x')\}_2 = -u_x\delta(x-x'),$$
$$\{\rho(x), u(x')\}_2 = u_x\delta(x-x'), \qquad \{\rho(x), \rho(x')\}_2 = -(\rho\partial_x + \partial_x\rho)\delta(x-x')$$

of the second Hamiltonian structure can be reduced by the Darboux theorem to the canonical form

$$\{w(x), w(x')\}_2 = \delta'(x-x'), \qquad \{\eta(x), w(x')\}_2 = 0,$$
$$\{w(x), \eta(x')\}_2 = 0, \qquad \{\eta(x), \eta(x')\}_2 = \delta'(x-x')$$

by the Miura-type transformation (see the first equation in (16))

$$w = u_x, \qquad \rho = -\tfrac{1}{2}(w^2 + \eta^2).$$

Then the Gurevich–Zybin system written in a *modified* form (see formula (26) in [7], other details in the last section 5 and references [6, 7] therein)

$$\eta_t + \partial_x(u\eta) = 0, \qquad u_{xt} + uu_{xx} + \tfrac{1}{2}u_x^2 = \tfrac{1}{2}\eta^2 \tag{28}$$

can be recognized as the two-component generalization of the Hunter–Saxton equation (cf [8–10]).

Remark 4. In fact, the Casimir density η of the second Hamiltonian structure was found in [5] (see formula (18) therein). However, the Gurevich–Zybin system was not presented in the form (28) there. Moreover, we emphasize the main result of this paper is that *the Gurevich–Zybin system belongs to the well-known class of integrable systems.* In this section we prove that the Gurevich–Zybin system is a member of an integrable hierarchy embedded into 2×2 spectral transform.

Since the first Poisson bracket in new field variables has the form

$$\{w(x), \eta(x')\}_1 = -\left[\frac{1}{\eta}\delta(x-x')\right]'', \qquad \{\eta(x), w(x')\}_1 = \frac{1}{\eta}\delta''(x-x'),$$
$$\{w(x), w(x')\}_1 = 0, \qquad \{\eta(x), \eta(x')\}_1 = -\left[\frac{w_x}{\eta^2}\partial_x + \partial_x\frac{w_x}{\eta^2}\right]\delta(x-x'),$$

then the modified Gurevich–Zybin system (28) as a member of an integrable hierarchy with all other commuting flows together can be written in the bi-Hamiltonian form

$$w_{t^k} = \partial_x\frac{\delta H_{k+1}}{\delta w} = -\partial_x^2\left[\frac{1}{\eta}\frac{\delta H_k}{\delta \eta}\right],$$
$$\eta_{t^k} = \partial_x\frac{\delta H_{k+1}}{\delta \eta} = \frac{1}{\eta}\partial_x^2\frac{\delta H_k}{\delta w} - \left[2\frac{w_x}{\eta^2}\partial_x + \left(\frac{w_x}{\eta^2}\right)_x\right]\frac{\delta H_k}{\delta \eta}.$$

An eigenvalue problem for the recursion operator as a ratio of both Hamiltonian structures

$$\begin{bmatrix} 0 & -\partial_x^2\frac{1}{\eta} \\ \frac{1}{\eta}\partial_x^2 & -\left(\frac{w_x}{\eta^2}\partial_x + \partial_x\frac{w_x}{\eta^2}\right)\end{bmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = 2\lambda\partial_x\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

can be written as one equation

$$\varphi_{xxx} + 4(\lambda^2\eta^2 + \lambda\sigma)\varphi_x + 2(\lambda^2\eta^2 + \lambda\sigma)_x\varphi = 0,$$

where $\varphi_1 = \varphi_x$, $\varphi_2 = -2\lambda\eta\varphi$ and $\sigma = w_x$. However, the above equation can be reduced to

$$\psi_{xx} + (\lambda^2\eta^2 + \lambda\sigma)\psi = 0,$$

where $\varphi = \psi\psi^+$ is a *squared eigenfunction* and ψ, ψ^+ are linear conjugate solutions with different asymptotics at infinity $\lambda \to \infty$. This linear spectral problem (more precisely, just 'x'-dynamics) is well known in the theory of integrable systems: corresponding systems are members (commuting flows) of the two-component Harry Dym hierarchy (see, for instance, [19]). All such members of this hierarchy can be determined by the spectral transform

$$\psi_{xx} = -(\lambda^2\eta^2 + \lambda\sigma)\psi, \qquad \psi_t = b\psi_x - \tfrac{1}{2}b_x\psi, \tag{29}$$

where $b(\zeta, \eta, \lambda)$ is a *polynomial* function with respect to the spectral parameter λ for *positive* members. The compatibility condition $(\psi_{xx})_t = (\psi_t)_{xx}$ yields the relationship

$$(\lambda^2\eta^2 + \lambda\sigma)_t = \left[\tfrac{1}{2}\partial_x^3 + 2(\lambda^2\eta^2 + \lambda\sigma)\partial_x + (\lambda^2\eta^2 + \lambda\sigma)_x\right] b,$$

where the two-component Harry Dym system (see [19])

$$\eta_{t_1} = \left(\frac{\sigma}{\eta^2}\right)_x, \qquad \sigma_{t_1} = \left(\frac{1}{\eta}\right)_{xxx}$$

can be obtained if $b = 2\lambda/\eta$. Thus, the *twice potential* two-component Harry Dym system

$$\eta_{t_1} = \left(\frac{u_{xx}}{\eta^2}\right)_x, \qquad u_{t_1} = \left(\frac{1}{\eta}\right)_x \tag{30}$$

is the *first* member of *positive* part of above hierarchy and the *first* member of its *negative* part is the *modified* Gurevich–Zybin system (28)

$$\eta_{t_{-1}} + \partial_x(u\eta) = 0, \qquad u_{xt_{-1}} + uu_{xx} + \tfrac{1}{2}u_x^2 = \tfrac{1}{2}\eta^2 \tag{31}$$

determined by the choice $b = (2\lambda)^{-1} - u$ (it means that we must identify $t \equiv t_{-1}$ for the Gurevich–Zybin system (2)).

Remark 5. The reciprocal transformation (see (8))

$$\mathrm{d}\tau_1 = \mathrm{d}t_1, \qquad \mathrm{d}\tau_{-1} = \mathrm{d}t_{-1}, \qquad \mathrm{d}z = \rho\,\mathrm{d}x - \rho u\,\mathrm{d}t_{-1} - \left(\frac{u_x}{\eta}\right)_x \mathrm{d}t_1$$

simultaneously linearizes the Gurevich–Zybin system (see (9) and (10)) and *preserves* the two-component Harry Dym system:

$$\rho_{t_1} = -\left(\frac{w}{\eta}\right)_{xx}, \qquad w_{t_1} = \left(\frac{1}{\eta}\right)_{xx}, \qquad \eta_{t_1} = \left(\frac{w_x}{\eta^2}\right)_x$$
$$\to \bar{\rho}_{\tau_1} = -\left(\frac{\bar{w}}{\bar{\eta}}\right)_{zz}, \qquad \bar{w}_{\tau_1} = \left(\frac{1}{\bar{\eta}}\right)_{zz}, \qquad \bar{\eta}_{\tau_1} = \left(\frac{\bar{w}_z}{\bar{\eta}^2}\right)_z,$$

where

$$\bar{\rho} = \frac{1}{\rho}, \qquad \bar{w} = -\frac{w}{\rho}, \qquad \bar{\eta} = \frac{\eta}{\rho}.$$

Such reciprocal *auto-transformation* is the first example in the theory of integrable systems.

Remark 6. The twice potential two-component Harry Dym system (30) written in field variables (ρ, u) was also found in [5] (see formula (21) therein), but was not *recognized*.

8. Reciprocal and Miura-type transformations

Application of the reciprocal transformation (in fact, it was given in [19], formulae (32)–(34) therein; cf (13))

$$\mathrm{d}y_1 = \mathrm{d}t_1, \qquad \mathrm{d}y_{-1} = \mathrm{d}t_{-1}, \qquad \mathrm{d}\zeta = \eta\,\mathrm{d}x - \eta u\,\mathrm{d}t_{-1} + \frac{u_{xx}}{\eta^2}\,\mathrm{d}t_1 \tag{32}$$

to the spectral transform (29) yields another well-known spectral transform (see, for instance, [19, 14]), where 'ζ'-dynamics is

$$\tilde{\psi}_{\zeta\zeta} + \left[\lambda^2 - \tilde{u}\lambda - \tilde{\upsilon} + \frac{\tilde{u}^2}{4}\right]\tilde{\psi} = 0, \tag{33}$$

'y'-dynamics is

$$\tilde{\psi}_{y_1} = (2\lambda + \tilde{u})\tilde{\psi}_\zeta - \frac{1}{2}\tilde{u}_\zeta\tilde{\psi}, \qquad \tilde{\psi}_{y_{-1}} = \frac{1}{4\lambda}(2\eta\tilde{\psi}_\zeta - \eta_\zeta\tilde{\psi}) \tag{34}$$

and

$$\tilde{\psi} = \eta^{1/2}\psi, \qquad -\tilde{u} = u_{\zeta\zeta} + \frac{\eta_\zeta}{\eta}u_\zeta, \qquad -\tilde{\upsilon} + \frac{\tilde{u}^2}{4} = \frac{\eta_\zeta^2}{4\eta^2} - \frac{\eta_{\zeta\zeta}}{2\eta}. \tag{35}$$

The compatibility conditions $(\tilde{\psi}_{\zeta\zeta})_{y_1} = \left(\tilde{\psi}_{y_1}\right)_{\zeta\zeta}$ and $(\tilde{\psi}_{\zeta\zeta})_{y_{-1}} = \left(\tilde{\psi}_{y_{-1}}\right)_{\zeta\zeta}$ yield the first *positive* member

$$\tilde{u}_{y_1} = 2\partial_\zeta\left[\frac{\tilde{u}^2}{2} + \tilde{\upsilon}\right], \qquad \tilde{\upsilon}_{y_1} = 2\partial_\zeta\left[\tilde{u}\tilde{\upsilon} - \frac{1}{4}\tilde{u}_{\zeta\zeta}\right], \tag{36}$$

and the first *negative* member

$$\tilde{u}_{y_{-1}} = -\eta_\zeta, \qquad \tilde{\upsilon}_{y_{-1}} = \frac{1}{2}\partial_\zeta(\tilde{u}\eta), \qquad -\frac{1}{2}\eta_{\zeta\zeta\zeta} + \left(2\tilde{\upsilon} - \frac{\tilde{u}^2}{2}\right)\eta_\zeta + \left(\tilde{\upsilon}_\zeta - \frac{1}{2}\tilde{u}\tilde{u}_\zeta\right)\eta = 0,$$

which also can be obtained by the limit $\tilde{\lambda} \to 0$ from the generating function of commuting flows

$$\begin{gathered}\tilde{u}_y = -\tilde{\eta}_\zeta, \qquad \tilde{\upsilon}_y = \partial_\zeta\left[\left(\frac{1}{2}\tilde{u} - \tilde{\lambda}\right)\tilde{\eta}\right], \\ \tilde{\eta}_{\zeta\zeta\zeta} + 4\left(\tilde{\lambda}^2 - \tilde{u}\tilde{\lambda} - \tilde{\upsilon} + \frac{\tilde{u}^2}{4}\right)\tilde{\eta}_\zeta + 2\left(-\tilde{\lambda}\tilde{u}_\zeta - \tilde{\upsilon}_\zeta + \frac{1}{2}\tilde{u}\tilde{u}_\zeta\right)\tilde{\eta} = 0,\end{gathered} \tag{37}$$

where $\tilde{\eta} = \tilde{\varphi}\tilde{\varphi}^+$ is a *squared eigenfunction* and $\tilde{\varphi}$, $\tilde{\varphi}^+$ are linear conjugate solutions of the spectral transform (33) with different asymptotics at infinity $\tilde{\lambda} \to \infty$ (for details see, for instance, [20]).

At the same time, as was proved in [7] (see also section 3 above), the modified Gurevich–Zybin system is *linearized* by the above reciprocal transformation (32). Simultaneously, the *twice potential* two-component Harry Dym system (30) transforms into the *twice degenerate twice modified* Kaup–Boussinesq system

$$u_{y_1} = -u_\zeta u_{\zeta\zeta} - \frac{\eta_\zeta}{\eta}\left(1 + u_\zeta^2\right), \qquad \eta_{y_1} = \eta u_{\zeta\zeta\zeta} + \left(\eta_{\zeta\zeta} - 2\frac{\eta_\zeta^2}{\eta}\right)u_\zeta. \tag{38}$$

It is well known that (36) is the Kaup–Boussinesq system (see, for instance, [21]). Several modified Kaup–Boussinesq systems were presented in [14]. The *modified* Kaup–Boussinesq system is

$$\tilde{u}_{y_1} = \partial_\zeta\left[\tfrac{3}{2}\tilde{u}^2 + 2u_1^2 + 2u_{1,\zeta}\right], \qquad u_{1,y_1} = \partial_\zeta\left[u_1\tilde{u} - \tfrac{1}{2}\tilde{u}_\zeta\right],$$

where u_1 is a new *intermediate* field variable and the *first* Miura transformation is

$$\tilde{\upsilon} = \tfrac{1}{4}\tilde{u}^2 + u_1^2 + u_{1,\zeta}. \tag{39}$$

The *twice modified* Kaup–Boussinesq system is

$$u_{1,y_1} = \partial_\zeta\left[\left(2u_1^2 - u_{1,\zeta}\right)u_2 - \tfrac{1}{2}u_{2,\zeta\zeta}\right], \qquad u_{2,y_1} = \partial_\zeta\left[2u_1\left(1+u_2^2\right) + u_2 u_{2,\zeta}\right],$$

where u_2 is another new *intermediate* field variable and the *second* Miura transformation is

$$\tilde{u} = 2u_1u_2 + u_{2,\zeta}. \tag{40}$$

It was proved in [14] that the Kaup–Boussinesq system has third and fourth Miura transformations (see also [22]). Their *double parametric degeneration* to *purely potential* form (cf (39), (40) with second and third equations from (35))

$$u_1 = \tfrac{1}{2}\partial_\zeta \ln\eta, \qquad u_2 = -u_\zeta \tag{41}$$

transforms the twice modified Kaup–Boussinesq system into the form (38).

Thus, the main result of this section is the establishment of the link of transformations (reciprocal and Miura type) between the Gurevich–Zybin and the Kaup–Boussinesq hierarchies.

9. Second harmonic generation

The generation of the second harmonic wave from the red light of a ruby laser in a crystal of quartz was in fact the starting point of nonlinear optics. In the one-dimensional case, for short pulses, when the group-velocity mismatch between both frequency components becomes important, then the process of second harmonic generation (SHG) is described by the complex equations (see, for instance, [23, 24]), which in real form are (see [24], formula (6) therein)

$$\tilde{u}_{y_{-1}} = -\eta_\zeta = -2\eta u_1, \qquad 2u_{1,y_{-1}} = -\frac{1}{\eta} + \eta\tilde{u}. \tag{42}$$

The corresponding spectral problem (see [24], formula (9) therein)

$$\tilde{\psi}_{\zeta\zeta} + \left[\lambda^2 - \tilde{u}\lambda - u_1^2 - u_{1,\zeta}\right]\tilde{\psi} = 0, \qquad \tilde{\psi}_{y_{-1}} = \frac{1}{2\lambda}\eta\tilde{\psi}_\zeta - \frac{1}{2\lambda}\eta u_1\tilde{\psi}$$

is a *special* reduction of the spectral transform (33), (34). In the general case (37) can be integrated once

$$\tilde{\eta}\tilde{\eta}_{\zeta\zeta} - \frac{1}{2}\tilde{\eta}_\zeta^2 + 2\left(\tilde{\lambda}^2 - \tilde{u}\tilde{\lambda} - \tilde{\upsilon} + \frac{\tilde{u}^2}{4}\right)\tilde{\eta}^2 + S(\tilde{\lambda}) = 0,$$

where $S(\tilde{\lambda})$ is a polynomial function for multi-periodic solutions of the Kaup–Boussinesq hierarchy (see for instance [25]). Thus, the first *negative* member of this hierarchy has the constraint

$$\eta\eta_{\zeta\zeta} - \frac{1}{2}\eta_\zeta^2 + 2\left(-\tilde{\upsilon} + \frac{\tilde{u}^2}{4}\right)\eta^2 + S_{-1} = 0,$$

where $S_{-1} \neq 0$ is some constant. However, $S_{-1} = 0$ in the case of the SHG system! It is easy to prove by direct substitution (39) and $\eta_\zeta = 2\eta u_1$ from (42) (see also the first equation in (41)) in the above equation. Thus, the SHG system (42) is the *degenerate first negative* member of the *modified* Kaup–Boussinesq hierarchy (see (39) and above).

The SHG system

$$\left(\ln \Xi_{y_{-1}}\right)_{\zeta y_{-1}} = \frac{1}{\Xi_{y_{-1}}} - \Xi_\zeta\,\Xi_{y_{-1}} \tag{43}$$

can be interpreted as the two-component generalization of the sinh-Gordon equation, where $\Xi_\zeta = \tilde{u}$ and $\Xi_{y_{-1}} = -\eta$. The SHG system has *three* different linearizable degenerations; as well as the sinh-Gordon equation it has a *parametric* degeneration to the famous Liouville equation which is linearizable. The first two degenerate limits are known (see [26, 27]). These are the Liouville equation and the modified Liouville equation. The *third* such case can be obtained by (see above) differential substitutions (40) and (41). This is the modified Gurevich–Zybin system (31) re-calculated by the reciprocal transformation (32) (see formula (27) in [7] and other details in the last section 5)

$$u_\zeta = \left(\frac{1}{\eta}\right)_{y_{-1}}, \qquad \eta_{y_{-1}y_{-1}} - \frac{3\eta_{y_{-1}}^2}{2\eta} + \frac{1}{2}\eta^3 = 0.$$

Thus, a solution of the *reduced* SHG system

$$\left(u_{\zeta\zeta} + \frac{\eta_\zeta}{\eta}u_\zeta\right)_{y_{-1}} = \eta_\zeta, \qquad (\ln\eta)_{\zeta y_{-1}} = -\frac{1}{\eta} - (\eta u_\zeta)_\zeta$$

in an implicit form is given by (17)

$$\eta = \left[\frac{1}{\eta_0(\zeta)} + \frac{\eta_0(\zeta)}{4}(y_{-1} - y_0(\zeta))^2\right]^{-1},$$

$$u = \frac{y_{-1}}{2}\int \eta_0(\zeta)\,\mathrm{d}\zeta - \frac{1}{2}\int \eta_0(\zeta)y_0(\zeta)\,\mathrm{d}\zeta,$$

$$x = \int\left[\frac{1}{\eta_0(\zeta)} + \eta_0(\zeta)y_0^2(\zeta)\right]\mathrm{d}\zeta - \frac{y_{-1}}{2}\int \eta_0(\zeta)y_0(\zeta)\,\mathrm{d}\zeta + \frac{y^2}{4}\int \eta_0(\zeta)\,\mathrm{d}\zeta.$$

Thus, a *new* solution of the SHG system (43) can be found in quadratures

$$\mathrm{d}\Xi = -\left(u_{\zeta\zeta} + \frac{\eta_\zeta}{\eta}u_\zeta\right)\mathrm{d}\zeta - \eta\,\mathrm{d}y_{-1}.$$

Final remark. Since the Kaup–Boussinesq system and the nonlinear Schrödinger equation are related by invertible transformations (see for instance [28]), then their first negative flows are also related. Since the first negative flow to the nonlinear Schrödinger equation is the famous Maxwell–Bloch system (in this particular case the 'self-induced transparency' coincides with the Maxwell–Bloch system), then the SHG system is connected to Maxwell–Bloch system by the same transformations. Thus, the Gurevich–Zybin system is connected with the Maxwell–Bloch system. Similarly, a new solution of the Maxwell–Bloch system can be found in the same way. Since the nonlinear Schrödinger equation relates to the Heisenberg magnet by Miura-type transformations, then a particular case of the 'Raman scattering' is also connected with the Maxwell–Bloch system. Thus, a new solution for the Raman scattering can be constructed as in the previous case.

10. Open problems

The numerical simulation of nonlinear dynamics described by the Gurevich–Zybin system yields the hypothesis that a behaviour in multimode form demonstrating the transition from the hydrodynamic to the equilibrium kinetic state has some regular features (see [1] for details) possibly generated by *integrable* properties of the corresponding N-component Gurevich–Zybin system (see (3))

$$\rho_t^k + \partial_x(\rho^k u^k) = 0, \qquad u_t^k + u^k u_x^k + \Phi_x = 0, \qquad \Phi_{xx} = \sum_{m=1}^{N}\rho^m.$$

This problem (integrability in any sense: linearization, inverse scattering transform, bi-Hamiltonian formulation, etc) is open. For instance, this system written in field variables u^k and υ^k $\left(\rho^k \equiv \upsilon^k_x\right)$ has the ultra-local Hamiltonian structure

$$\upsilon^k_t = \frac{\delta H_2}{\delta u^k}, \qquad u^k_t = -\frac{\delta H_2}{\delta \upsilon^k},$$

where the Hamiltonian is

$$H_2 = \frac{1}{2}\int \left[-\sum_{m=1}^{N} (u^m)^2 \upsilon^m_x + \left(\sum_{m=1}^{N} \upsilon^m\right)^2 \right] \mathrm{d}x.$$

It was proved here that the Gurevich–Zybin system has infinitely many Hamiltonian structures. Existence of the second Hamiltonian structure is enough for an integrability.

Introducing moments

$$A_k = \sum_{m=1}^{N} (u^m)^k \rho^m \tag{44}$$

then the N-component Gurevich–Zybin system can be written as the nonlocal chain

$$\partial_t A_k + \partial_x A_{k+1} + k A_{k-1} \partial_x^{-1} A_0 = 0, \qquad k = 0, 1, 2, \ldots, \tag{45}$$

which looks very similar to the famous *integrable* Benney moment chain (see [29])

$$\partial_t A_k + \partial_x A_{k+1} + k A_{k-1} \partial_x A_0 = 0, \qquad k = 0, 1, 2, \ldots.$$

The Benney moment chain has infinitely many N-component reductions (see [30]) parametrized by N functions of a single variable (N is an arbitrary natural integer), where the simplest reduction is (44). The nonlocal chain (45) has at least one simple reduction

$$A_k = \rho u^k.$$

The existence of any other such reductions could be a *symptom* of integrability. The integrability of N-component Gurevich–Zybin system and a description of other reductions of the nonlocal chain (45) will be considered elsewhere.

11. Conclusion

The Gurevich–Zybin system is an example of integrable systems possessing properties of two different classes: C- and S-integrable. This system is linearizable and has a general solution. Thus, the Gurevich–Zybin system is from a C-integrable class. However, this system has an infinite set of Hamiltonian structures and commuting flows. Thus, the Gurevich–Zybin system is also from a S-integrable class. Moreover, this system has an infinite set of *local* Hamiltonian structures, which is unusual in the theory of S-integrable systems. Moreover, all commuting flows of the Gurevich–Zybin system written in the form of a Monge–Ampere equation are the same Monge–Ampere equation again. The difference between them is just some derivative of the function $\mu(z)$, which can be eliminated by a point transformation $\upsilon = \mu'''(z)$ (see (7)). Thus, this is a beautiful example from a mathematical point of view having a physical application (see [1]). Let us emphasize again: *every commuting flow* can be written in the *same form* (23)

$$\Phi_{xx}\Phi_{t^k t^k} - \Phi^2_{xt^k} = K(\Phi_x)\Phi_{xx},$$

where $K(z)$ is any *a priori* given function, but all commuting flows will be written via different functions $\Phi_{(k)}$, because the point transformation like $\upsilon = \mu'''(z)$ becomes nonlocal

$$K(\Phi_{(k)x}) = \mu^{(2k+2)}(\Phi_x).$$

The exceptional case is when some derivative $\mu^{(n)}(z)$ is a constant: then 'half' the commuting flows are trivial (see [16, 31])

$$\Phi_{xx}\Phi_{t^k t^k} - \Phi^2_{xt^k} = 0,$$

when n is even, then $k \geqslant n/2$, when n is odd, then $k \geqslant (n-1)/2$. In the last 10 years a couple of such examples of integrable systems (mixed properties of C- and S-integrability) were found in [16, 32]. However, an explanation of such phenomena does not exist at the moment. One possible explanation is that such systems are an *intersection* of C- and S-integrability. Thus, they *accumulate* properties of these two different classes.

Moreover, we proved that the Gurevich–Zybin system is a *degenerate* member of the two-component Harry Dym hierarchy. A *degeneracy* arises when equations can possess a *parametric* freedom. When some of parameters are fixed (to zero, for instance), then such equations become *linearizable*. The simplest example is the famous Liouville equation

$$w_{xt} = \mathrm{e}^{w}.$$

This equation is an *intersection* of two different integrable hierarchies. Another of them is the another famous Bonnet equation (well known in physics as the sinh-Gordon equation) first introduced in the differential geometry of surfaces of constant curvature

$$w_{xt} = c_1\,\mathrm{e}^{w} + c_2\,\mathrm{e}^{-w},$$

which is a member of the potential *modified* KdV hierarchy (spectral transform 2×2)

$$w_\tau = w_{xxx} - \tfrac{1}{2}w_x^3.$$

Another one is the Tzitzeica equation, well known in affine differential geometry

$$w_{xt} = c_1\,\mathrm{e}^{w} + c_2\,\mathrm{e}^{-2w},$$

which is a member of the potential *modified* Sawada–Kotera hierarchy (spectral transform 3×3)

$$w_\tau = w_{xxxxx} + 5\left(w_{xx}w_{xxx} - w_x^2 w_{xxx} - w_x w_{xx}^2\right) + w_x^5.$$

Thus, if $c_2 = 0$, then the Liouville equation is still *a member of two different integrable hierarchies simultaneously*. This is a good *symptom* that such equations should be *linearizable*. Since the above-mentioned linearizable reduction of the SHG system is determined by 2×2 spectral transform, but the SHG system is some reduction of another important three-wave interaction problem (see, for instance, [24]), then we can assume such systems as the Gurevich–Zybin system are linearizable if they are an intersection of at least two different integrable hierarchies (e.g. the Liouville equation is a member of two different hierarchies: of the KdV equation and of the Sawada–Kotera equation). The bi-Hamiltonian structure presented here has its origin in the 2×2 spectral transform. It will be interesting to find another Hamiltonian structure coming from a 3×3 spectral problem.

Finally, we would like to emphasize that this paper was devoted to the recognition of the relationship between a couple of remarkable systems having applications in astrophysics, nonlinear optics and geometry.

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